Review of Vector Geometry

Vector

A quantity that has magnitude and direction. Examples of quantities that can be represented by vectors include:

Displacements – traverse lines, GPS baselines, navigation sectors, tunnel sections
Movement – velocity, acceleration
Dynamics – force, gravity, momentum

Free Vectors

In general, vectors have no specific location or point of application – thus in a geomatics context, vectors are regarded as "free"

Fixed Vectors

In many geomatics applications, a vector has a particular location and is therefore termed "fixed" or "bound". In a plane surveying context, an example of a fixed vector is the "position vector" which gives the location of a point with respect to a datum and as such is merely another way of giving the "coordinates" of the point.

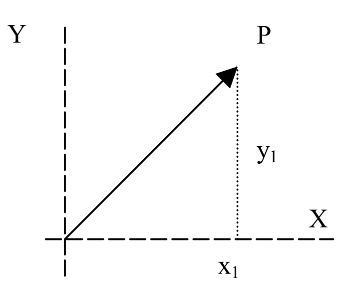


Figure 1

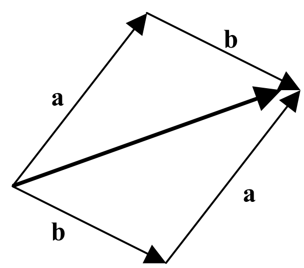


Figure 2

In Figure 1, the point P is located by the position vector $\mathbf{p} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$

Line of Action

the line of infinite length along which a "sliding" vector is free to move – a sliding vector being a partially fixed vector.

Vector Addition

is represented by the geometrical construction in Figure 2. In surveying terms the vectors **a** and **b** represent "movements" or "displacements". When the displacement **a** is added to the displacement **b** the result is the displacement **c** .

Obviously; $\mathbf{c} = \mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$

Vector Components

The vectors **a** and **b** in Figure 2 can be though of as "components" of vector **c** . Obviously there can be any number of components of a vector – the vector sum of the components giving the total vector.

Thus in Figure 3: $\mathbf{c} = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4 + \mathbf{v}_5 + \mathbf{v}_6$

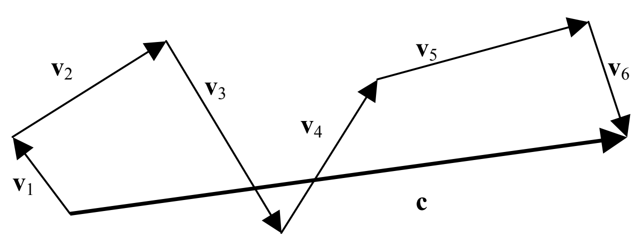


Figure 3

Orthogonal Components

Any vector can be represented by a set of mutually perpendicular – known as "orthogonal components". Obviously, 2-dimensional vectors can have only 2 orthogonal components and 3-dimensional vectors, 3 orthogonal components in any set. In Figure 4, the vectors **x** and **y** are the orthogonal components of the vector **p** in the reference frame XY. The vectors **x'** and **y'** are the orthogonal components of the vector **p** in the reference frame X'Y'.

Figure 5 shows the 3-dimensional components of the vector **p** .

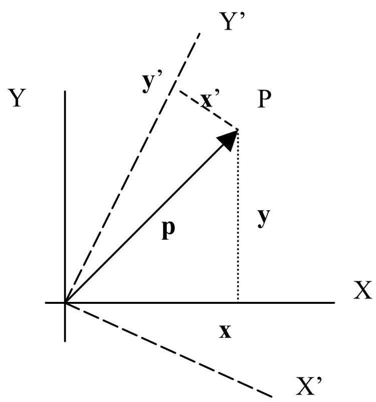


Figure 4

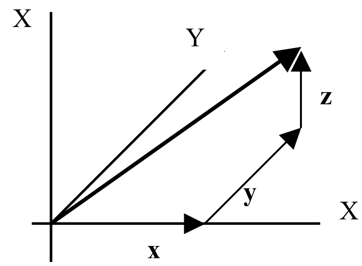


Figure 5

In the vector algebra used in these lectures, "component" will always refer to orthogonal components as these are fundamental to applications in surveying.

Unit Vectors

vectors with a length of unity.

Unit orthogonal vectors

a set of orthogonal cartesian axes are conveniently defined by unit orthogonal vectors – in the 3-dimensional case these are commonly represented by the unit vectors **i** , **j** , **k** ,

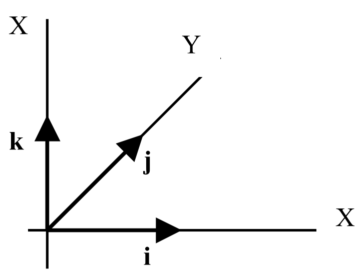


Figure 6

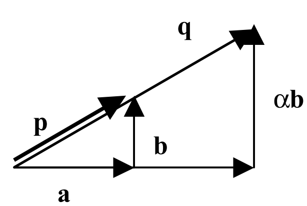


Figure 7

Scalar Multiplication

Scalar multiplication is demonstrated in Figure 7 where the vector **p** is magnified by the factor α .

Thus: $\mathbf{q} = \alpha\,\mathbf{p}$

If the vectors **a** and **b** are components of **p** , $\alpha\mathbf{a}$ and $\alpha\mathbf{b}$ are components of **q** .

The last equation infers that:

(1) Every vector can be represented by the sum of two scaled, co-planar linearly independent vectors
(2) Two linearly independent (ie non-parallel) vectors are sufficient to "span" 2-dimensional space.

Evaluating Vector Equations

Figure 8 represents a closed vector figure made up of scalar multiples of the vectors **a** and **b** . As moving from point A to point B can be done in two ways the vector equation can be written:

$$\alpha_1\mathbf{a} + \beta_1\mathbf{b} + \alpha_2\mathbf{a} + \beta_2\mathbf{b} + \alpha_3\mathbf{a} + \beta_3\mathbf{b} = \alpha\mathbf{a} + \beta\mathbf{b}$$

(movement through the higher path) = (movement through the lower path)

The diagram shows that the scalar coefficients can be equated as:

$$\alpha_1 + \alpha_2 + \alpha_3 = \alpha$$
$$\beta_1 + \beta_2 + \beta_3 = \beta$$

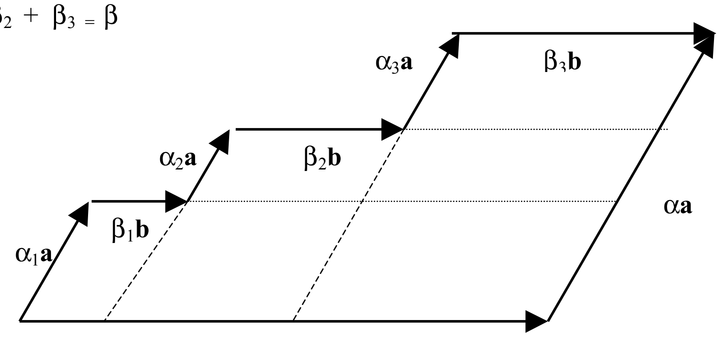


Figure 8

Linearly Independent Vectors

For the parallel vectors **a** and **b** (see Figure 9) the following equation is valid.

$\alpha\mathbf{a} = \beta\mathbf{b}$ (for α, β not zero)

In this case, vectors **a** and **b** are linearly dependent.

For the non-parallel vectors **c** and **d** the following equation is NOT valid.

$\alpha\mathbf{c} = \beta\mathbf{d}$ (for α, β not zero)

In this case, vectors **c** and **d** are linearly independent.

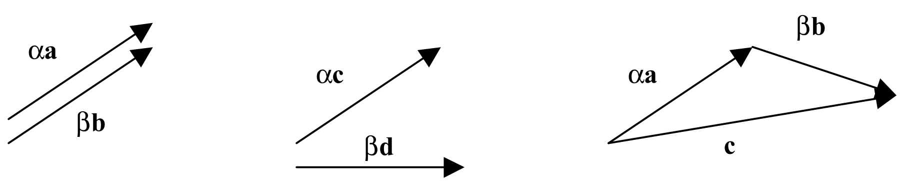


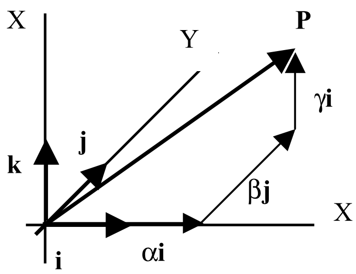
Figure 9

In Figure 9, if vector **c** is co-planar with the linearly independent vectors **a** and **b** , the following equation is valid:

$\mathbf{c} = \mathbf{a} + \mathbf{b}$

That is, **c** is linearly dependent on **a** and **b** .

Vector representation – Orthogonal Components or Coordinates ?



The "position vector" **p** of the Point P , has orthogonal components α**i**, β**j**, γ**k** – these are scalar multiples of the set of orthogonal unit vectors **i** , **j** , **k** which define the 3-dimensional Cartesian axis system X , Y, Z . The scalar multipliers are obviously the coordinates of the point P.

Summary of vector equations

(1) Commutative Law: $\mathbf{p} + \mathbf{q} = \mathbf{q} + \mathbf{p}$

(2) Associative Law: $(\mathbf{p} + \mathbf{q}) + \mathbf{r} = \mathbf{p} + (\mathbf{q} + \mathbf{r})$

(3) Commutative Law: $\alpha\,\mathbf{p} = \mathbf{p}\,\alpha$

(4) Associative Law $\alpha\,(\beta\,\mathbf{p}) = \beta\,(\alpha\,\mathbf{p}) = (\alpha\beta)\,\mathbf{p}$

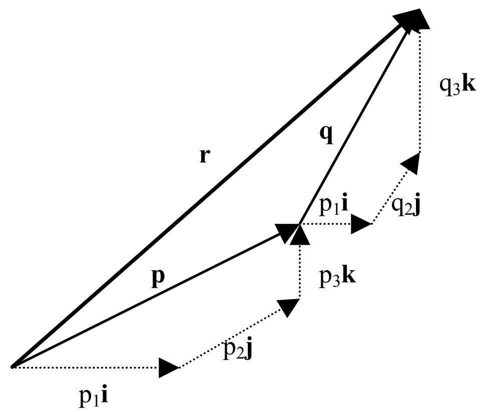
(5) Distributative Law: $(\alpha + \beta)\,\mathbf{p} = \alpha\,\mathbf{p} + \beta\,\mathbf{p}$

(6) Distributative Law: $\alpha\,(\mathbf{p} + \mathbf{q}) = \alpha\,\mathbf{p} + \alpha\,\mathbf{q}$

(7) Notation: $\alpha\,\mathbf{p} = \alpha \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$

Addition of Vector Components

The vector addtion rule can be extended to include vector components.



$$\mathbf{p} = p_1\,\mathbf{i} + p_2\,\mathbf{j} + p_3\,\mathbf{k}$$
$$\mathbf{q} = q_1\,\mathbf{i} + q_2\,\mathbf{j} + q_3\,\mathbf{k}$$

Hence:

$$\mathbf{r} = \mathbf{p} + \mathbf{q}$$
$$\mathbf{r} = (p_1 + q_1)\,\mathbf{i} + (p_2 + q_2)\,\mathbf{j} + (p_3 + q_3)\,\mathbf{k}$$

Preferred Notation

The preferred notation for this course is to write vectors as:

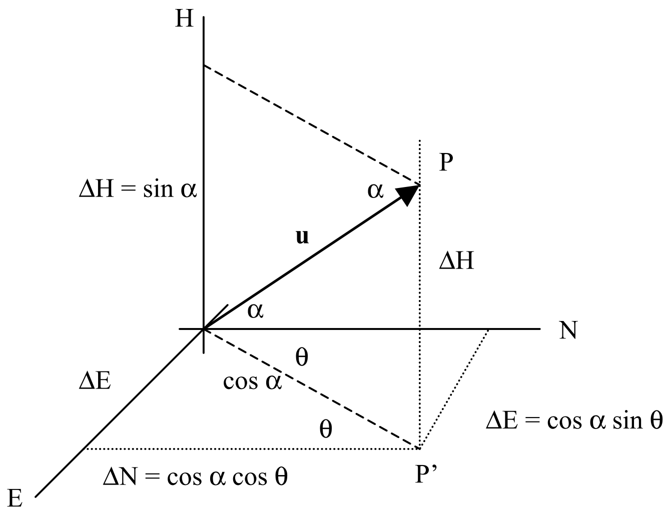
$$\mathbf{p} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \qquad \mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} \qquad \mathbf{r} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}$$

Hence the equation: $\mathbf{r} = \mathbf{p} + \mathbf{q}$

Would be written as:

$$\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} + \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} p_1 + q_1 \\ p_2 + q_2 \\ p_3 + q_3 \end{bmatrix}$$

The unit vector defining the line of sight of a theodolite/total station within an East, North, Height axis system.



u is the unit vector defining the direction of the line of sight
θ is the whole circle bearing of the line of sight
α is the elevation of the line of sight, positive above the horizon, negative below the horizon

Formulae to remember:

$$\mathbf{u} = \begin{bmatrix} \Delta E \\ \Delta N \\ \Delta H \end{bmatrix} = \begin{bmatrix} \cos\alpha\sin\theta \\ \cos\alpha\cos\theta \\ \sin\alpha \end{bmatrix}$$

$$\frac{\Delta E}{\Delta N} = \frac{\cos\alpha\sin\theta}{\cos\alpha\cos\theta} = \tan\theta$$

$$\Delta H = \sin\alpha$$

Therefore,

$$\theta = a\tan\left(\frac{\Delta E}{\Delta N}\right)$$

$$\alpha = a\sin(\Delta H)$$

For 2D-problems α = 0

$$\mathbf{u} = \begin{bmatrix} \Delta E \\ \Delta N \end{bmatrix} = \begin{bmatrix} \sin\theta \\ \cos\theta \end{bmatrix}$$

$$\theta = a\tan\left(\frac{\Delta E}{\Delta N}\right)$$

The Dot Product

The unit vector u of the general vector v

If **v** is the vector (in an East, North, Height) axis system,

$$\mathbf{v} = \begin{bmatrix} \Delta e \\ \Delta n \\ \Delta h \end{bmatrix}$$

its magnitude is given as: $|\,\mathbf{v}\,| = [\,\Delta e^2 + \Delta n^2 + \Delta h^2\,]^{1/2}$

so that its unit vector: $|\,\mathbf{u}\,| = \dfrac{\mathbf{v}}{|\,\mathbf{v}\,|}$

or alternatively: $|\,\mathbf{u}\,| = \dfrac{1}{|\,\mathbf{v}\,|}\begin{bmatrix} \Delta e \\ \Delta n \\ \Delta h \end{bmatrix} = \begin{bmatrix} \Delta e / |\,\mathbf{v}\,| \\ \Delta n / |\,\mathbf{v}\,| \\ \Delta h / |\,\mathbf{v}\,| \end{bmatrix}$

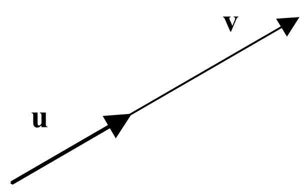


Figure 1

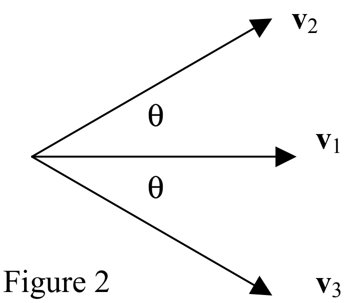


Figure 2

The Dot Product - Definition

$$\mathbf{v}_1 \bullet \mathbf{v}_2 = |\,\mathbf{v}_1\,|\;|\,\mathbf{v}_2\,|\;\cos\theta = \mathbf{v}_2 \bullet \mathbf{v}_1$$

$$\mathbf{v}_1 \bullet \mathbf{v}_3 = |\,\mathbf{v}_1\,|\;|\,\mathbf{v}_3\,|\;\cos\theta = \mathbf{v}_3 \bullet \mathbf{v}_1$$

If $|\,\mathbf{v}_1\,| = |\,\mathbf{v}_2\,|$, then $\mathbf{v}_1 \bullet \mathbf{v}_2 = \mathbf{v}_1 \bullet \mathbf{v}_3 = \cos\theta$

This last statement shows that if the dot product is used to determine the angle between two vectors, it cannot distinguish between a positive or negative angle.