Relationship between point and plane

The vector normal to a plane

The plane in Figure 2 is defined by the three points A, B, C whose position vectors (coordinates) are known. The two vectors $\mathbf{v}_1$ (displacement AB) and $\mathbf{v}_2$ (displacement AC) can be determined. If $\mathbf{p}_a$, $\mathbf{p}_b$, $\mathbf{p}_c$ are the position vectors of A, B, C, then

$$\mathbf{v}_1 = \mathbf{p}_b - \mathbf{p}_a \qquad\qquad \mathbf{v}_2 = \mathbf{p}_c - \mathbf{p}_a$$

Remember that that position vector is the term given to the vector which locates a point with respect to some datum or axis system. The components of a position vector are the coordinates of the point (see Figure 1).

The vector giving the displacement between two points is given by the difference between the two position vectors of the points. That is, the vector from P to Q is $(\boldsymbol{q} - \boldsymbol{p})$

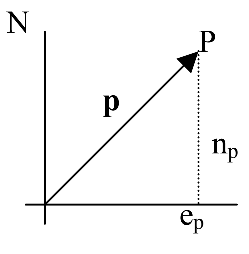
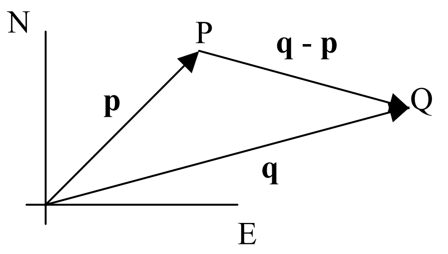


Figure 1



A vector normal to the plane is given by: $\mathbf{n} = \mathbf{v}_1 * \mathbf{v}_2$

The unit normal vector to the plane is: $\tilde{\mathbf{n}} = \dfrac{\mathbf{n}}{|\mathbf{n}|}$

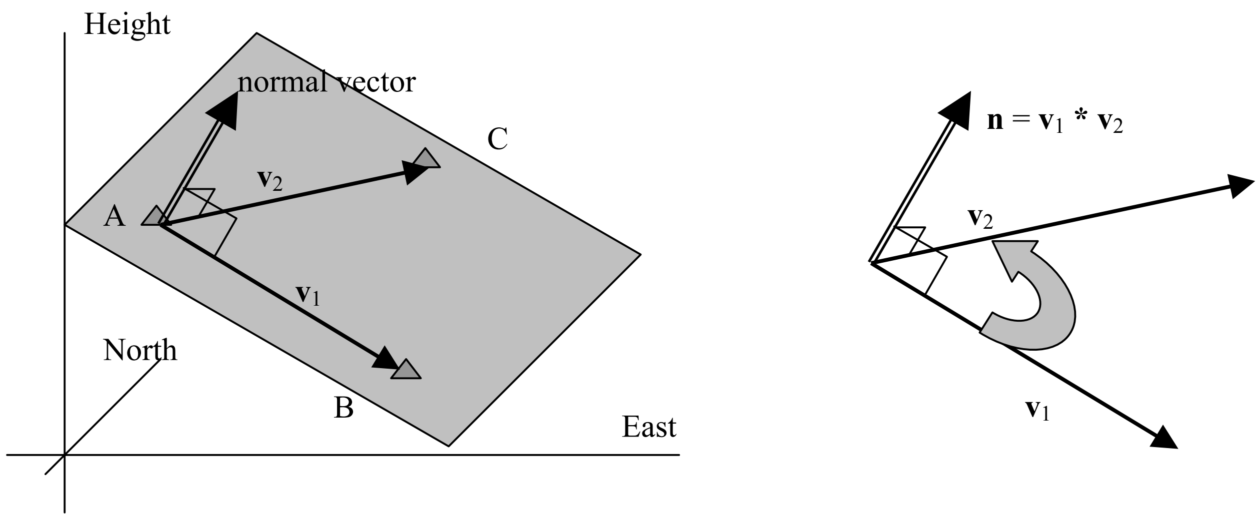


Figure 2

In Figure 3, P is a point, with known position vector outside the plane ABC. The first problem is to find the position vector of the point Q in the plane closest to P . The line PQ will obviously be at. right angles to the plane ABC . That is, the line QP will be in the direction of the unit normal vector $\tilde{\mathbf{n}}$ to the plane.

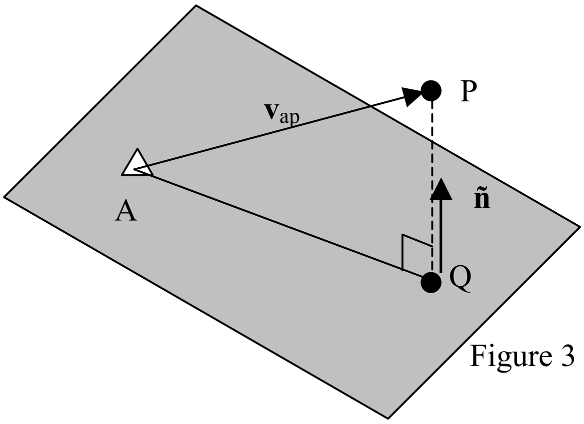
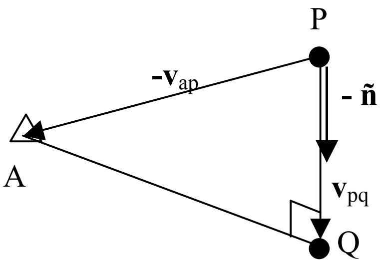


Figure 3



$PQ = -\tilde{\mathbf{n}} \cdot -\mathbf{v}_{ap} = \tilde{\mathbf{n}} \cdot \mathbf{v}_{ap}$
$\mathbf{v}_{pq} = (\tilde{\mathbf{n}} \cdot \mathbf{v}_{ap})\, -\tilde{\mathbf{n}}$

The distance PQ is given by the dot product: $-\tilde{\mathbf{n}} \cdot -\mathbf{v}_{ap} = \tilde{\mathbf{n}} \cdot \mathbf{v}_{ap}$
(from the definition of the dot product - the projection of $-\mathbf{v}_{ap}$ onto the direction defined by $-\tilde{\mathbf{n}}$.

Thus the vector $\quad \mathbf{v}_{pq} = (\tilde{\mathbf{n}} \cdot \mathbf{v}_{ap}) - \tilde{\mathbf{n}} = -(\tilde{\mathbf{n}} \cdot \mathbf{v}_{ap})\, \tilde{\mathbf{n}}$
and the position vector of Q: $\quad \mathbf{p}_q = \mathbf{p}_a + \mathbf{v}_{ap} - (\tilde{\mathbf{n}} \cdot \mathbf{v}_{ap})\, \tilde{\mathbf{n}}$

Notice that in the example above the point P was on the side of the plane to which the normal vector was pointing. We should consider the case when P is on the other side of the plane or "below" the plane – as shown in Figure 4.

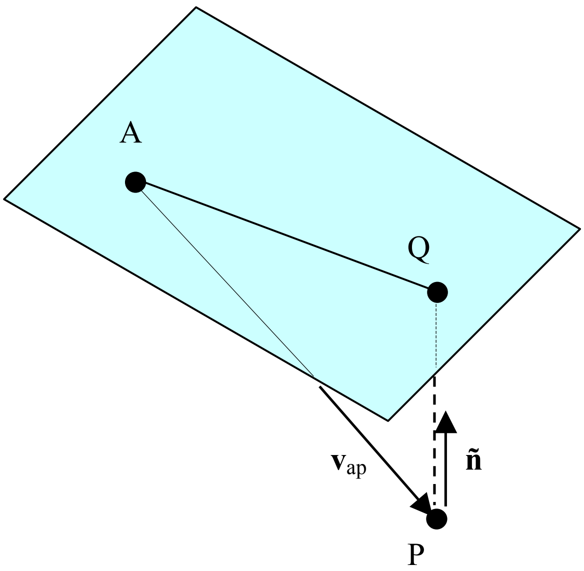


Figure 4

In this case he distance PQ is given by the dot product but now using $+\tilde{\mathbf{n}}$

$$PQ = \tilde{\mathbf{n}} \cdot -\mathbf{v}_{ap} = -\tilde{\mathbf{n}} \cdot \mathbf{v}_{ap}$$

Actually, this fact can be used to tell whether P is on the side to which the normal points.

The rule is:
- the shortest distance from a point to a plane is $(\tilde{\mathbf{n}} \cdot \mathbf{v}_{ap})$
- if $(\tilde{\mathbf{n}} \cdot \mathbf{v}_{ap})$ is positive, the point is on the side of the plane to which the normal vector points.

The intersection of a vector through a point adjacent to a plane and the plane.

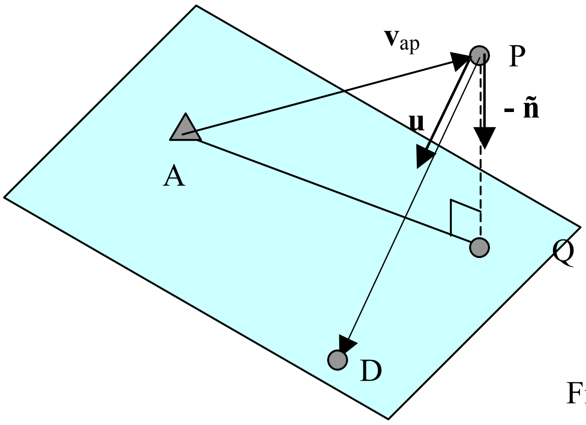


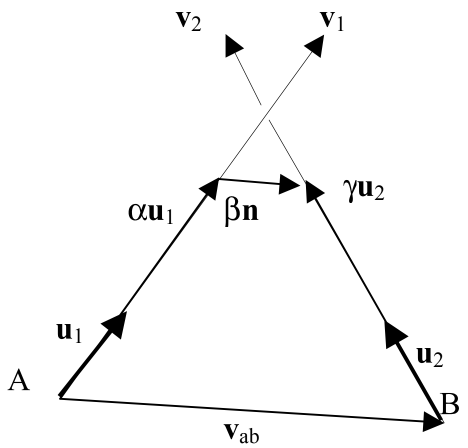
Figure 5

$\mathbf{u}$ ~ unit vector defining the direction of a line through P
D ~ point of intersection of line and plane
$\mathbf{n}$ ~ unit vector normal to the plane
$\mathbf{p}_p$, $\mathbf{p}_a$ ~ known postion vectors of points P and A .
$\mathbf{v}_{pa}$ ~ vector giving displacement PA = $\mathbf{p}_a - \mathbf{p}_p$
$\alpha\mathbf{u}$ ~ the vector giving displacement PD .
α ~ unknown scalar multiple.

The distance PQ = $\mathbf{\tilde{n}} \cdot \mathbf{v}_{ap}$ and
The distance PQ = $-\mathbf{\tilde{n}} \cdot \alpha\mathbf{u} = \alpha\, -\mathbf{\tilde{n}} \cdot \mathbf{u}$

Hence: $\boxed{\alpha = (\mathbf{\tilde{n}} \cdot \mathbf{v}_{ap}) / (-\mathbf{\tilde{n}} \cdot \mathbf{u})}$

$\boxed{\mathbf{p}_d = \mathbf{p}_p + \alpha\mathbf{u}}$



Shortest distance between two vectors

In Figure 6,
$\mathbf{v}_1$, $\mathbf{v}_2$ ~ vectors through points A and B whose position vectors are known.
$\mathbf{u}_1$, $\mathbf{u}_2$ ~ unit vectors of $\mathbf{v}_1$, $\mathbf{v}_2$
$\mathbf{v}_3$ ~ vector at right angles to both $\mathbf{v}_1$ and $\mathbf{v}_2$
$\mathbf{n}$ ~ unit vectors of $\mathbf{v}_3$ – normal to both $\mathbf{u}_1$ and $\mathbf{u}_2$

The shortest distance must be along the vector at right angles to both $\mathbf{v}_1$ and $\mathbf{v}_2$

Thus the vector $\mathbf{v}_3$ in the direction of the shortest distance will be $\mathbf{v}_1 * \mathbf{v}_2$

Thus the vector equation can be written:

$$\mathbf{V}_{ab} = \alpha\, \mathbf{u}_1 + \beta\, \mathbf{n} + \gamma\, \mathbf{u}_2$$

and solved for α, β and γ.